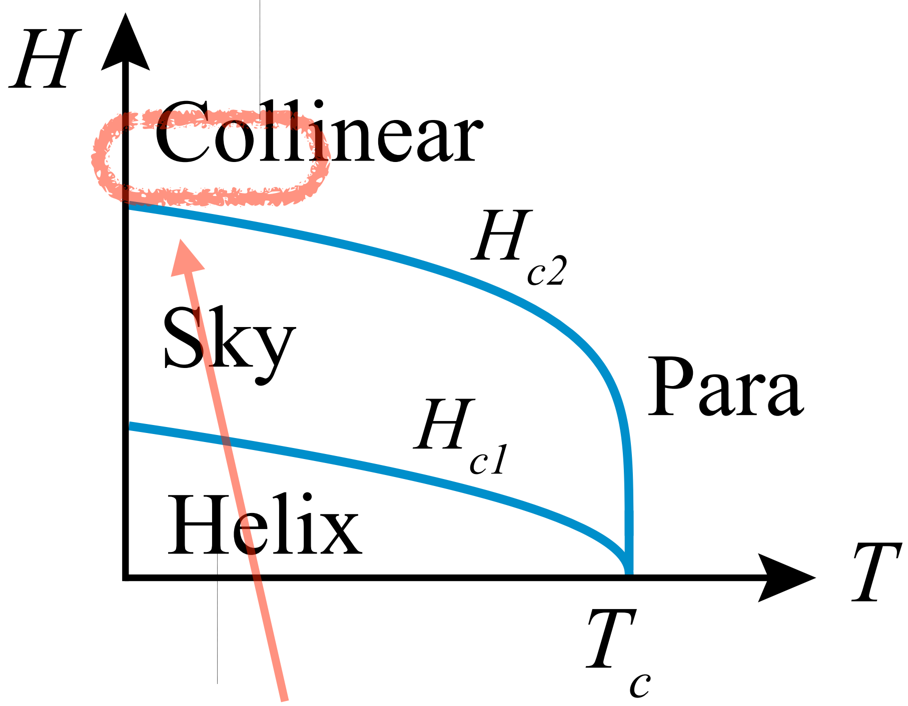
Semiclassical dynamics and Hall effects of quantum skyrmions

Héctor Ochoa

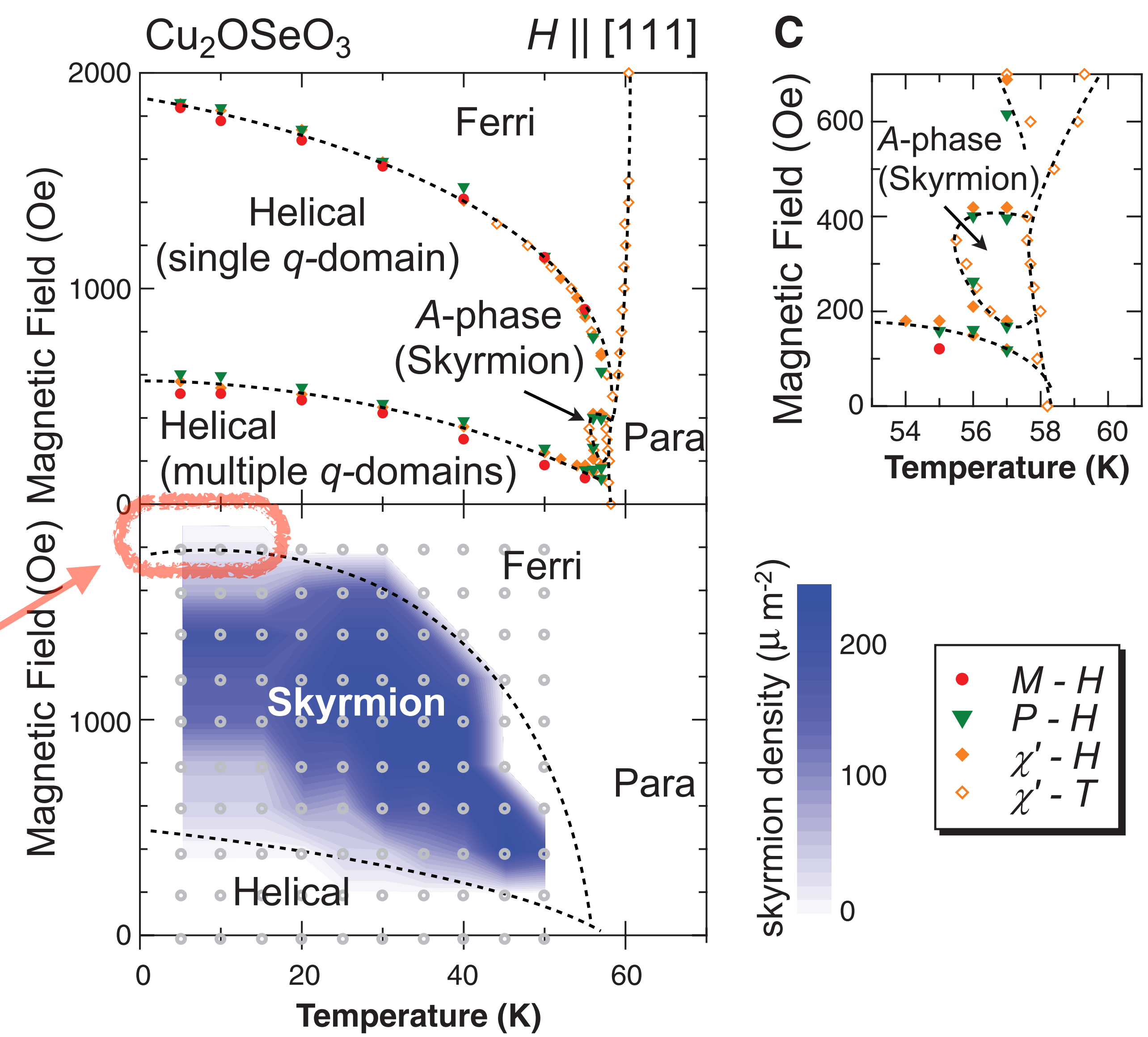
University of California, Los Angeles

In collaboration with Yaroslav Tserkovnyak (UCLA)
(to appear as a RMP colloquium soon)

Skyrmions as quasiparticles



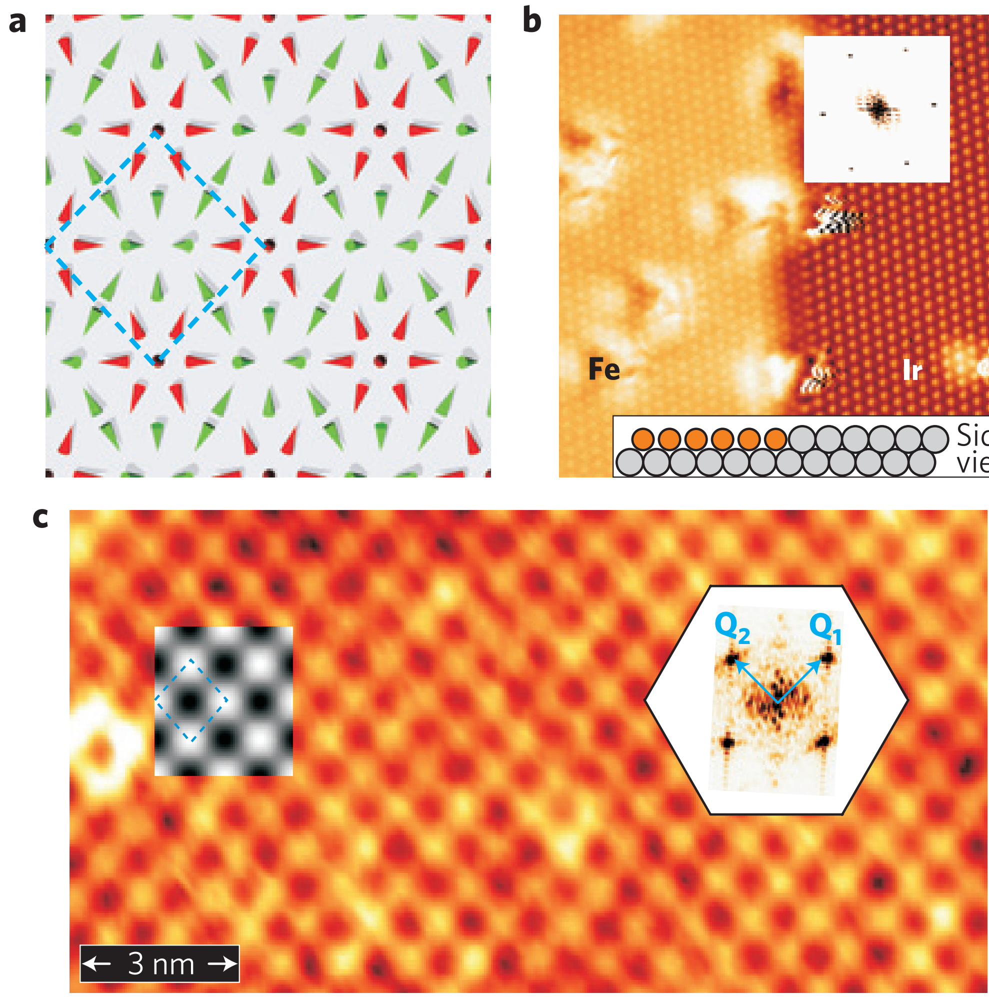
skyrmions are the lowest energy excitations



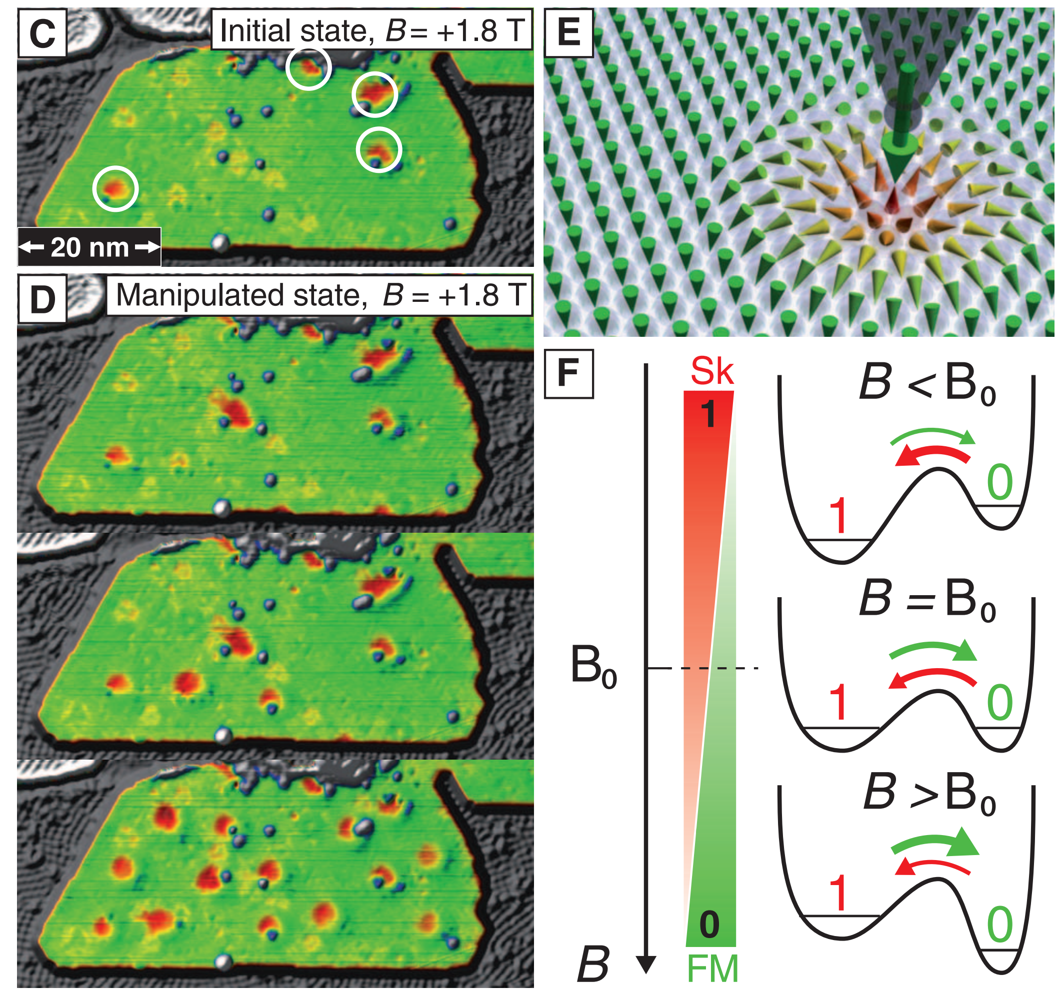
The analysis is restricted to planar chiral (ferro-)magnets

Seki *et al.*, Science **336**, 198 (2012)

Quantum-size limit



Heinze *et al*., Nat. Phys. **7**, 713 (2011)



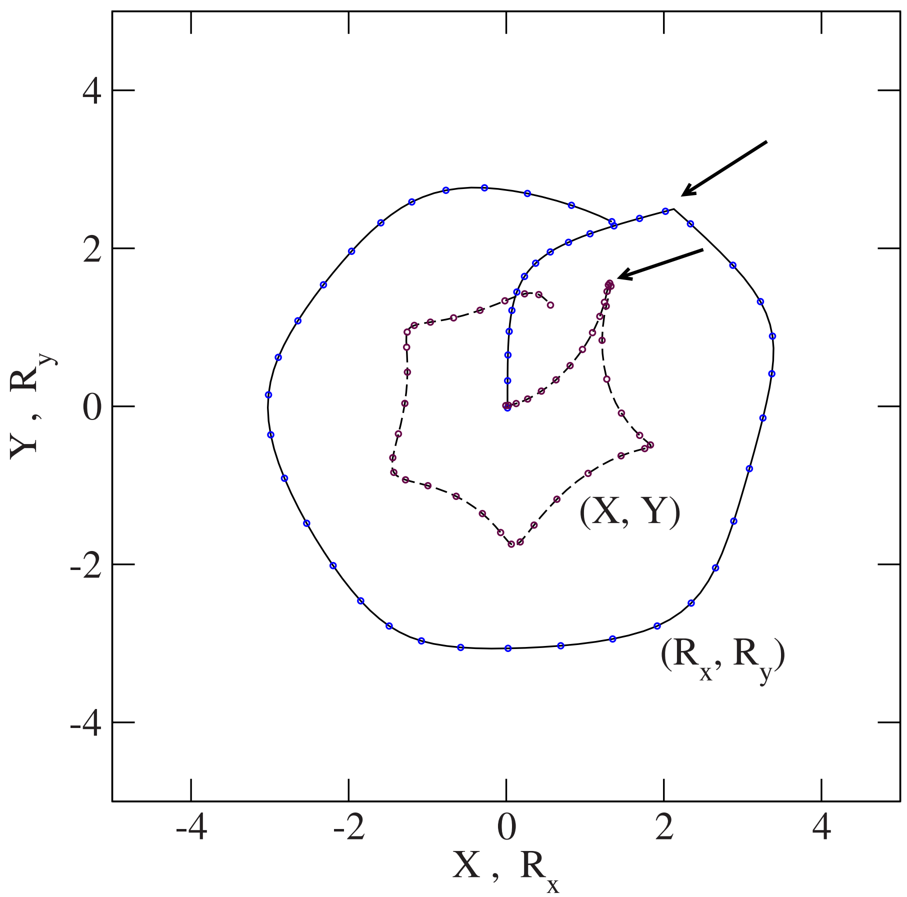
Romming *et al*., Science **341**, 636 (2013)

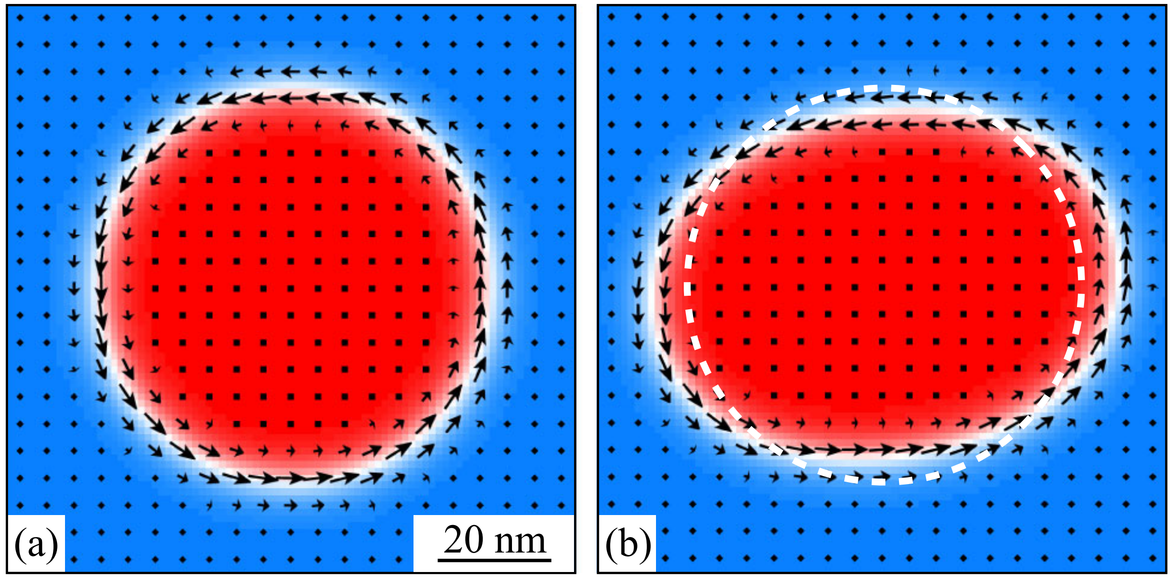
Questions Which are the skyrmion quantum numbers? How is the spectrum?

How is the Magnus force manifested in this regime?

Semiclassical quantization: basic idea

Moutafis *et al.*, PRB **79**, 224429 (2012)





Makhfudz *et al.*, PRL **109**, 217201 (2012)

Micromagnetic simulations
in confined geometries

The radius of *cyclotron orbits* caused by the lattice potential characterize the extension of quantum fluctuation in the phase space of *rigid* skyrmions

Outline

- Landau-Lifshitz dynamics
- Semiclassical quantization of collective coordinates
- Skyrmion bands and Berry phases
- Wave-packet dynamics and thermal Hall effect

Landau-Lifshitz equation: Hamiltonian formulation

At temperatures much lower than T_c:

$$\dot{\boldsymbol{s}}\left(\mathbf{r}\right)=\boldsymbol{s}\left(\mathbf{r}\right)\times\underline{\boldsymbol{h}_{\mathrm{eff}}\left(\mathbf{r}\right)}$$

Free-energy functional

conjugate force: $\boldsymbol{h}_{\mathrm{eff}}\equiv-\dfrac{\delta H}{\delta\boldsymbol{s}}$

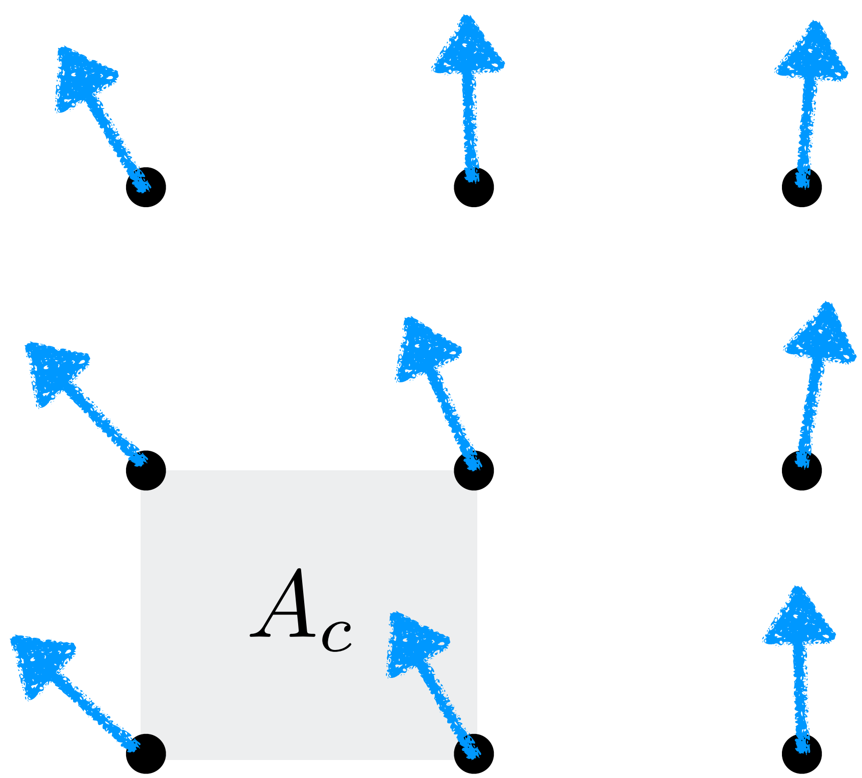
The LL eq can be recast as a Liouville equation of the form:

$$\dot{\boldsymbol{s}}\left(\mathbf{r}\right)=\left\{\boldsymbol{s}\left(\mathbf{r}\right),H\right\}=\int d^2\mathbf{r}'\left\{\boldsymbol{s}\left(\mathbf{r}\right),s_\alpha\left(\mathbf{r}'\right)\right\}\frac{\delta H}{\delta s_\alpha\left(\mathbf{r}'\right)}$$

generated by the Poisson algebra: $\boxed{\left\{s_\alpha\left(\mathbf{r}\right),s_\beta\left(\mathbf{r}'\right)\right\}=\epsilon_{\alpha\beta\gamma}\,s_\gamma\left(\mathbf{r}\right)\delta\left(\mathbf{r}-\mathbf{r}'\right)}$

2 complementary perspectives

- **Spin-hydrodynamics**: kinematic constrain on the classical phase-space
- Coarse-graining from **quantum** operators defined on a lattice:

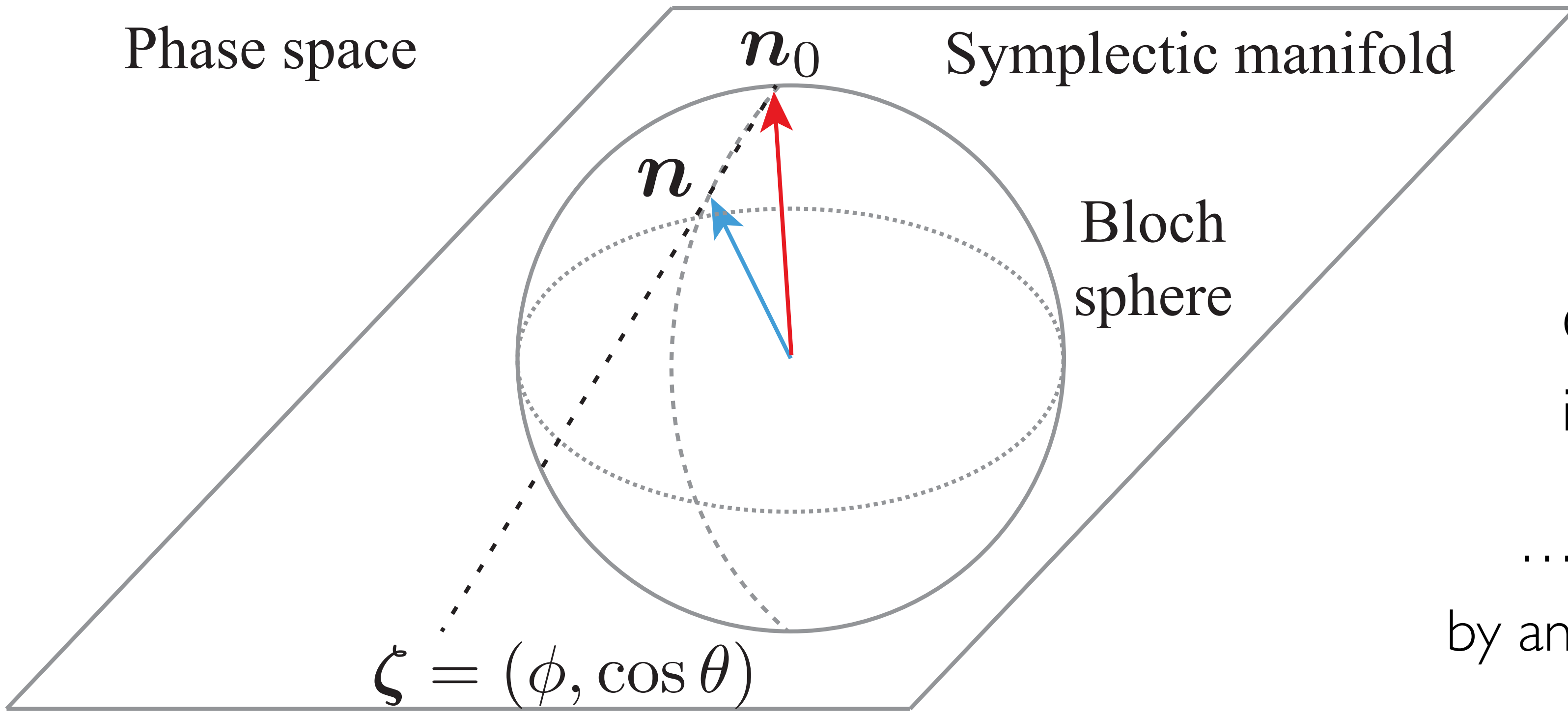


$$\boldsymbol{s}\left(\mathbf{R}_i\right) \approx \left\langle \Psi_{\rm sc} \right| \hat{\boldsymbol{S}}_i \left| \Psi_{\rm sc} \right\rangle / A_c$$

$$\{\,,\} \longrightarrow -\frac{i}{\hbar}\,[\,,]$$

we recover the quantum spin dynamics

Symplectic reduction



Non-canonical system

Canonical variables are (globally) ill-defined for generic textures…

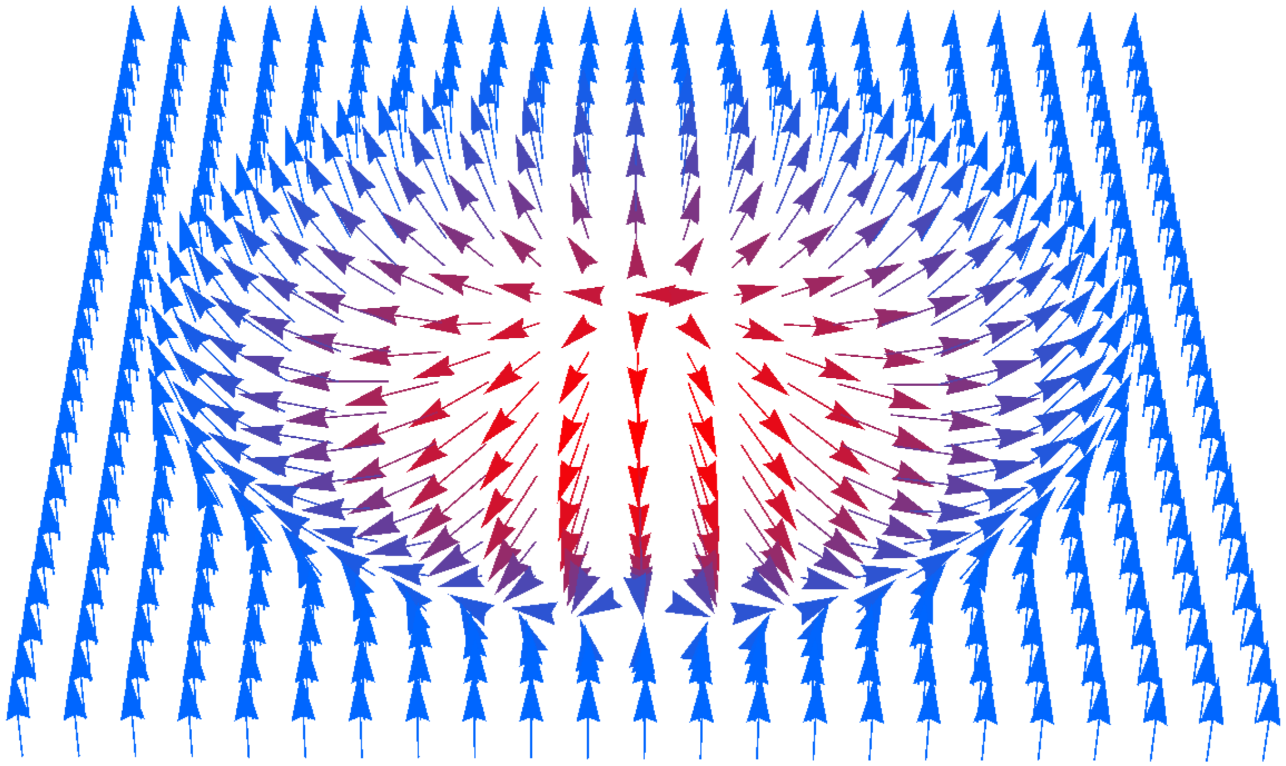
…but not for textures characterized by an integer winding number: **skyrmions**

$$L = \int d\mathbf{r}\ \underbrace{\boldsymbol{A}\left[\zeta\left(\mathbf{r}\right)\right]}_{4\pi s-\text{jumps}} \cdot \dot{\zeta}\left(\mathbf{r}\right) - H = s\int d\mathbf{r}\ \underbrace{\boldsymbol{a}\left[\boldsymbol{n}\left(\mathbf{r}\right)\right]}_{\boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{a} = -\boldsymbol{n}\ \text{(monopole field)}} \cdot \dot{\boldsymbol{n}}\left(\mathbf{r}\right) - H$$

$$\boxed{s = \hbar S/A_c}$$

spin-path quantization

Skyrmion textures: collective coordinates



$$\mathcal{Q} = \frac{1}{4\pi} \int d\mathbf{r}\, \boldsymbol{n} \cdot (\partial_x \boldsymbol{n} \times \partial_y \boldsymbol{n})$$

The number of times that the texture sweeps the singular point is fixed by the boundary conditions

$\boldsymbol{R} = (X, Y) \equiv$ center (1st moment) of topological charge

The skyrmion position is soft mode of the magnetization dynamics:

$$\{X, Y\} = \frac{1}{4\pi s \mathcal{Q}}$$

Linear momentum of *rigid* skyrmions

Canonical momentum: $\mathbf{\Pi} = 4\pi s \mathcal{Q}\, \boldsymbol{R} \times \hat{\boldsymbol{z}} \Longrightarrow \{R_i, \Pi_j\} = \delta_{ij}$

$\mathbf{\Pi}$ is the generator of rigid translations of the texture:

$$\{\Pi_i, \boldsymbol{s}\,[\boldsymbol{R}]\} = -\frac{\partial \boldsymbol{s}\,[\boldsymbol{R}]}{\partial R_i}$$

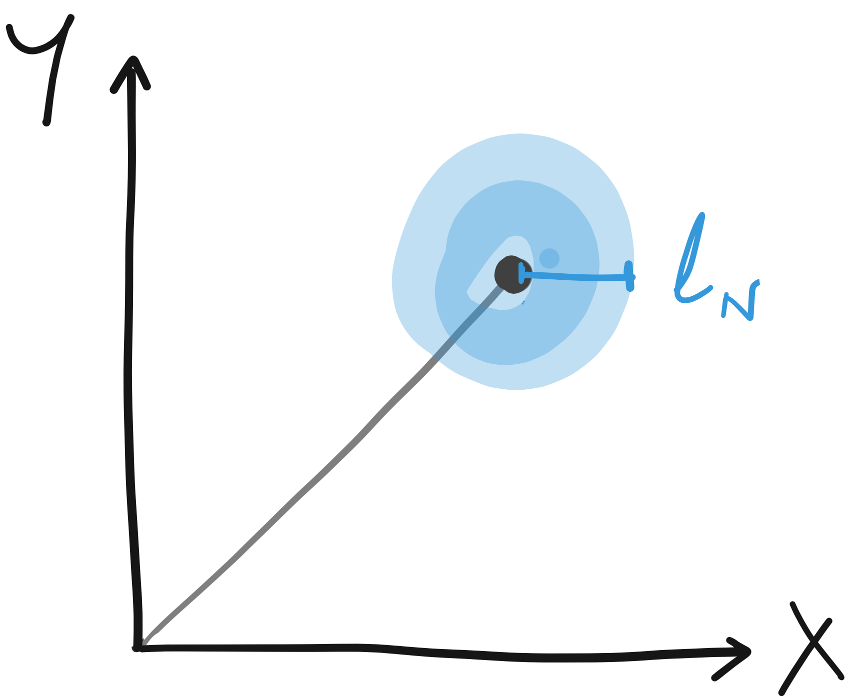
The Magnus force is a manifestation of the *jumps* in the functional-generator of translations:

$$\{\Pi_i, \Pi_j\} = \underbrace{G_{ij} = 4\pi s \mathcal{Q}\, \epsilon_{ij}}_{\text{gyrotropic tensor}}$$

Canonical quantization

$$\{X, Y\} \longrightarrow \left[\hat{X}, \hat{Y}\right] = \pm \frac{i\left(\ell_N\right)^2}{2}$$

$\mathrm{sign}\,(\mathcal{Q})$

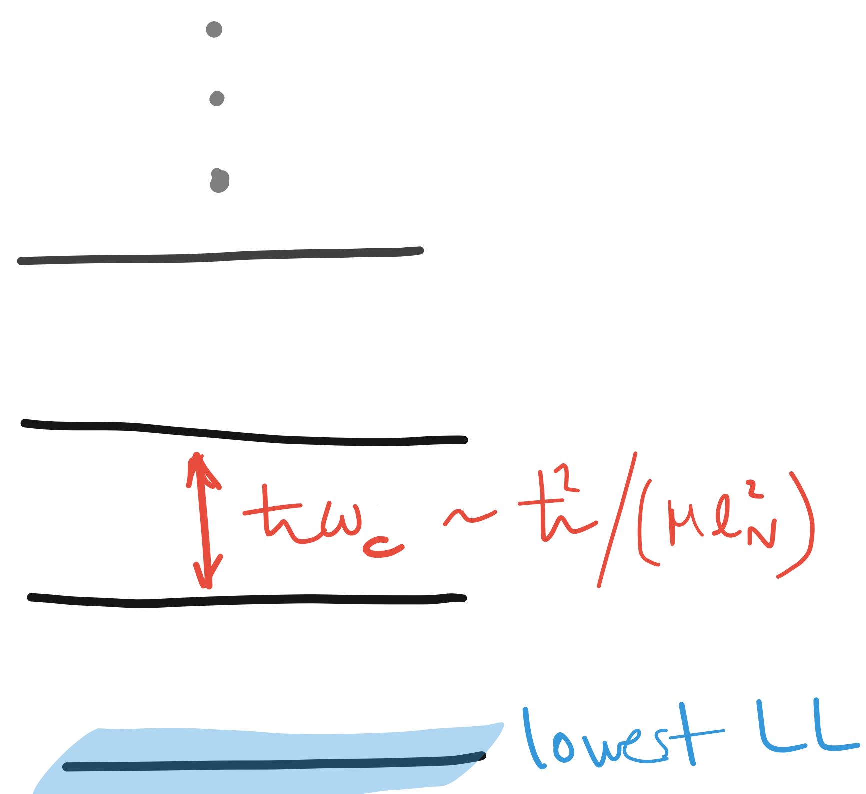


$$\ell_N \equiv \sqrt{\frac{A_c}{\pi N}} \quad \text{with} \quad N = 2S\,|\mathcal{Q}|$$

Classical limit: $\ell_N \ll \sqrt{A_c} \equiv S \to \infty$

The *quantum cloud* $A_c^* = \pi\left(\ell_N\right)^2$ is commensurate with the lattice

Truncation of the Hilbert space



$$\underline{M \to 0} \quad \hbar\omega_c \to \infty$$

skyrmion mass due to the hybridization with gapped modes

Dimension of the Hilbert space:

$$\frac{N_c \times A_c}{A_c^*} = N \times \underline{N_c}$$

unit cells

$$\frac{1}{2}\boldsymbol{Q} \wedge \boldsymbol{\Pi} \longrightarrow \hat{L}_z = \mp\hbar\left(\hat{a}^\dagger\hat{a} + \boxed{\frac{1}{2}}\right)$$

Aharonov-Bohm phase around the Dirac string

A semi-classical skyrmion is a coherent superposition of magnon-bound states

Lattice Hamiltonian

The translation symmetry is reduced to the group of lattice translations

Classically: $V\left(\boldsymbol{R}\right) = \sum_{\mathbf{G}} V_{\mathbf{G}}\, e^{i\mathbf{G}\cdot\boldsymbol{R}}$ with $V_{\mathbf{G}} = \int d\mathbf{r}\, \underbrace{\mathcal{H}_{\text{sky}}\left(\mathbf{r}\right)}_{\text{free-energy density of the classical solution}}\, e^{i\mathbf{G}\cdot\mathbf{r}}$

$\mathbf{G} = 0 \Longrightarrow \varepsilon_0 \equiv$ energy of the classical solution

$$\mathbf{G} \neq 0 \Longrightarrow V_{\mathbf{G}} \sim \frac{\varepsilon_0}{\left(R\left|\mathbf{G}\right|\right)^{3/2}}$$

the Fourier harmonics decay algebraically with the skyrmion radius

Translation operators

$$\sum_{\mathbf{G}} V_{\mathbf{G}}\, e^{i\mathbf{G}\cdot\boldsymbol{R}} \longrightarrow \sum_{\mathbf{G}} V_{\mathbf{G}}\, \underset{\displaystyle e^{i\mathbf{G}\cdot\hat{\boldsymbol{R}}}}{\underset{|||}{\hat{T}\left(\mathbf{G}\right)}}$$

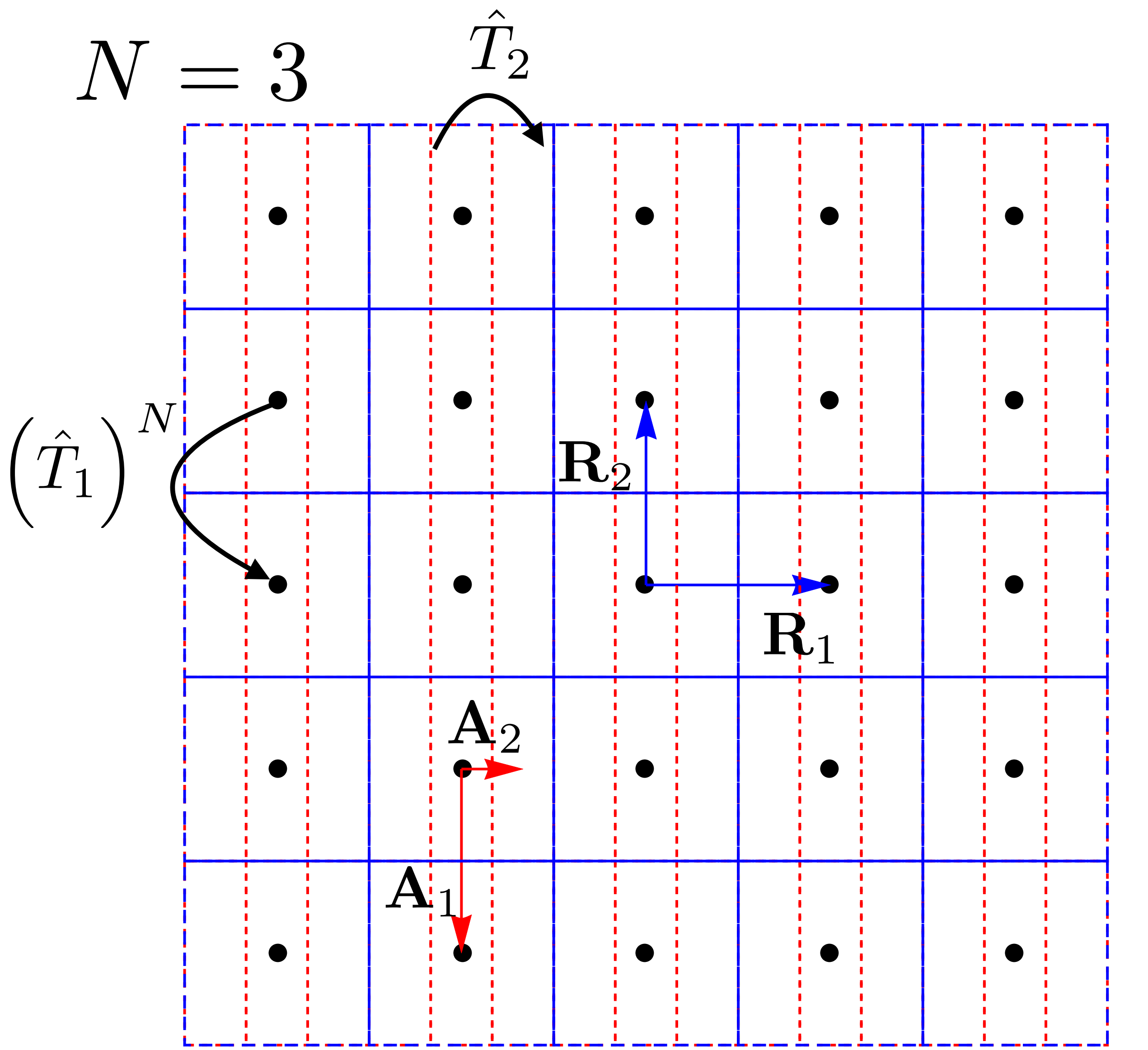
The translation operators form a *ray* group, like the *magnetic translations*:

$$\hat{T}\left(\mathbf{G}\right)\hat{T}\left(\mathbf{G}'\right) = \boxed{\mathrm{Exp}\left[\frac{\pm i\left(\mathbf{G}\wedge\mathbf{G}'\right)A_c}{4\pi N}\right]}\, T\left(\mathbf{G}+\mathbf{G}'\right)$$

$N = 2S\left|\mathcal{Q}\right|$ *quantum fluxes* per unit cell

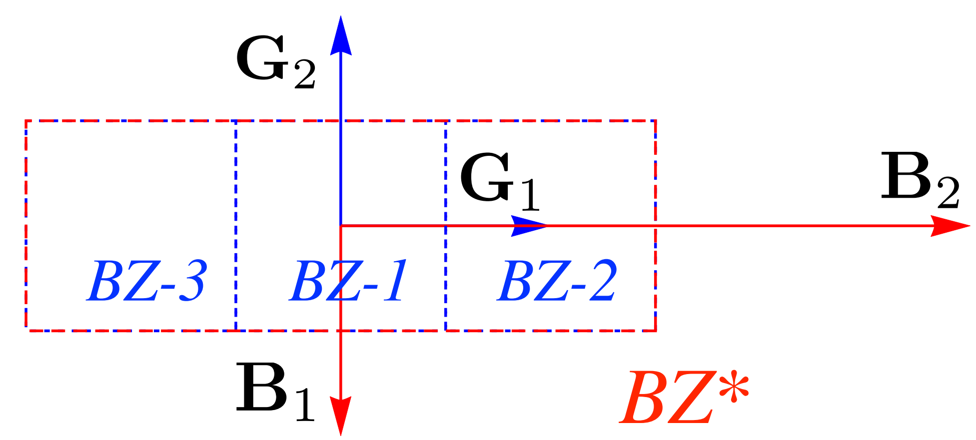
The lattice translational symmetry of the Hamiltonian is preserved

von Neumann lattice and Bloch states



Spin lattice von Neumann lattice

Reciprocal space (Brillouin zones)



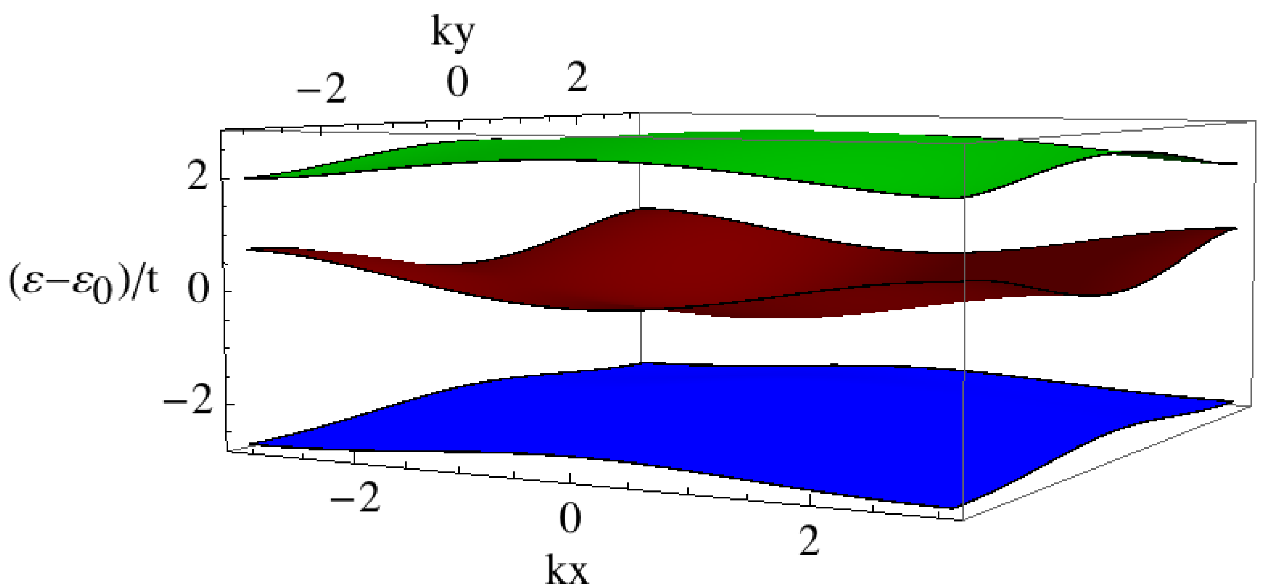
Bloch states: simultaneous eigenstates of

$$\left(\hat{T}_1\right)^N \quad \hat{T}_2$$

provide a $N-dim$ irrep of translations:

$$\hat{T}_{n,m} = e^{\mp \frac{inm\pi}{N}} \left(\hat{T}_1\right)^n \left(\hat{T}_2\right)^m$$

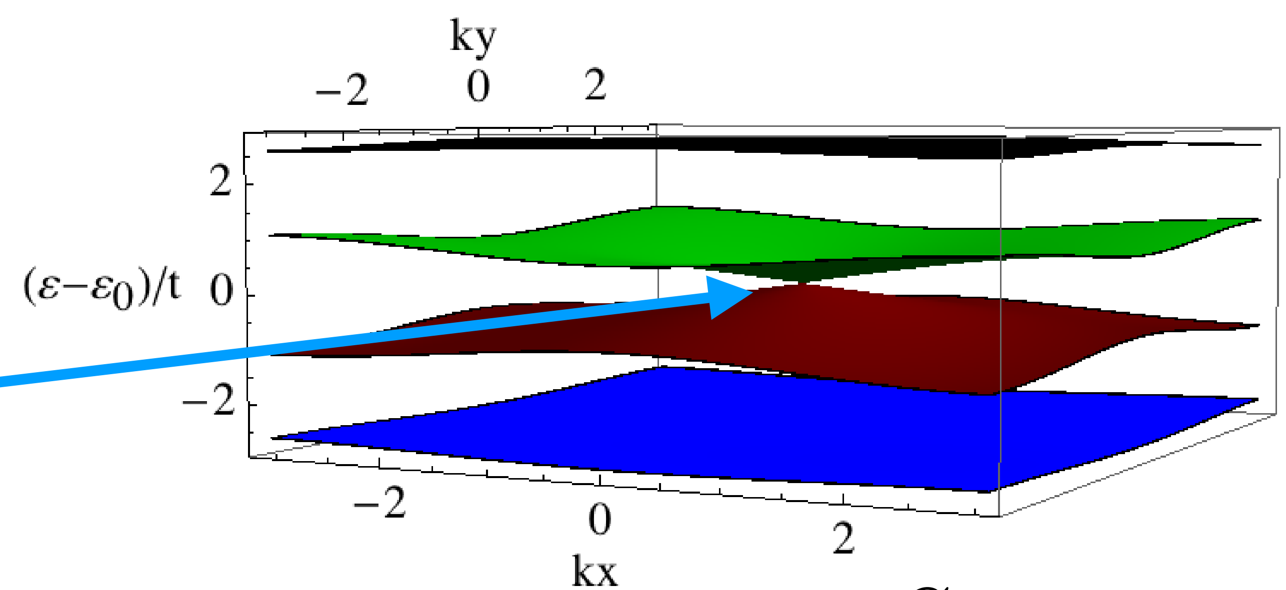
Skyrmion bands in the square lattice



Only first harmonics ***t***

$$|\mathcal{Q}| = 1$$

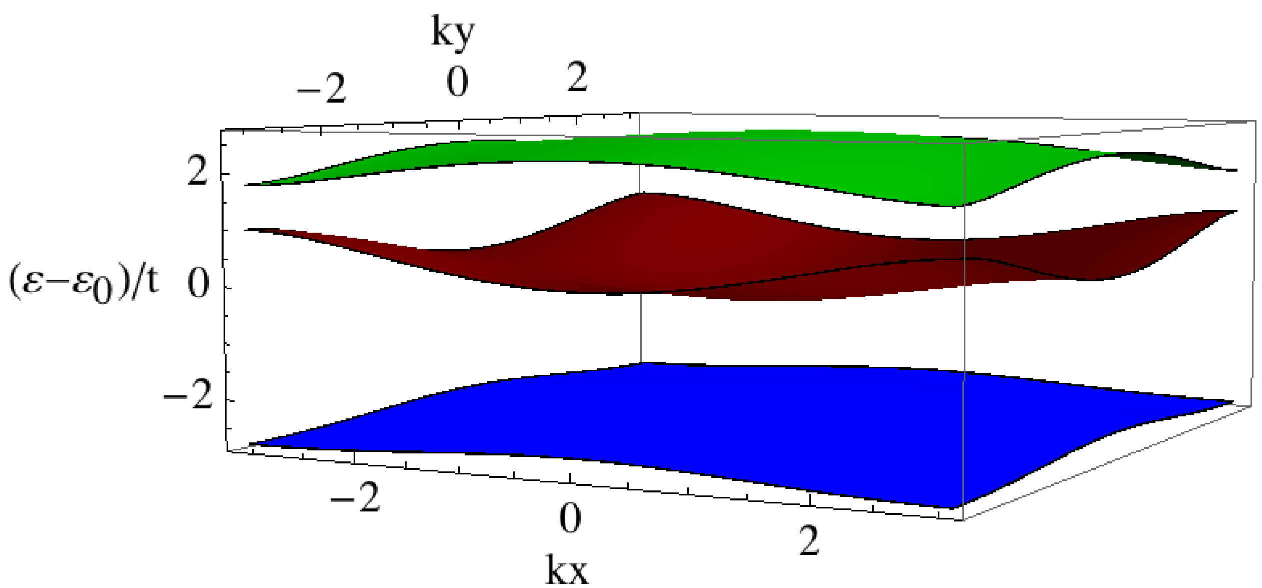
$S = 3/2$



Dirac points
for integer spins

$S = 2$

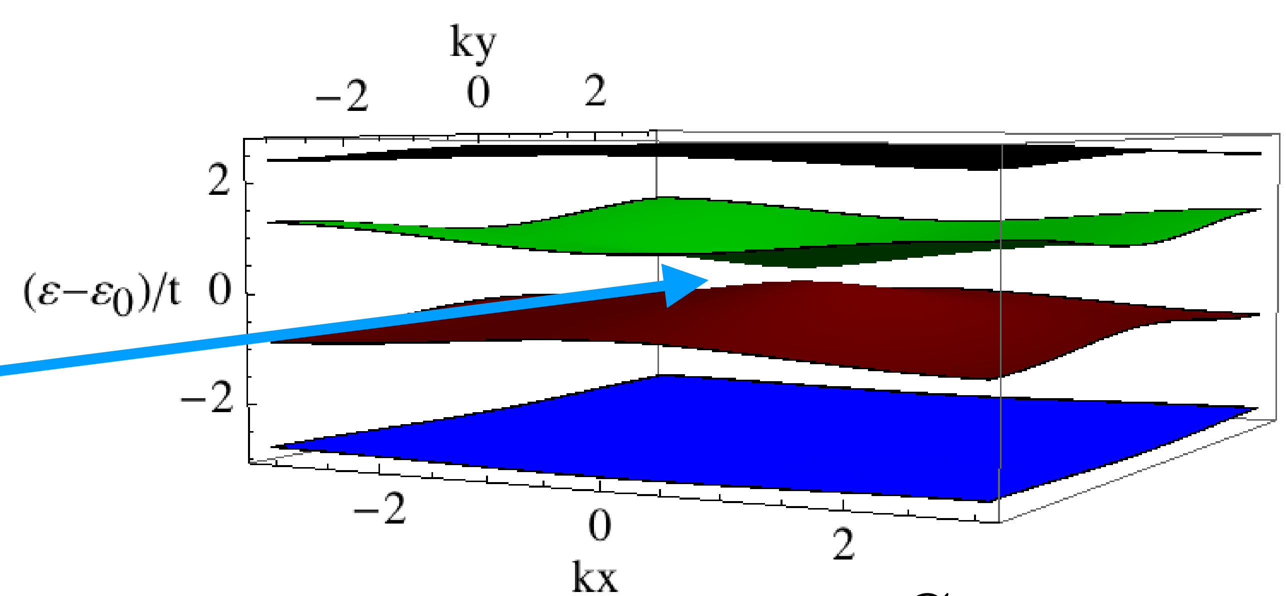
Skyrmion bands in the square lattice



Including 2nd harmonics ***t'***

$$|\mathcal{Q}| = 1$$

$$S = 3/2$$



Gapped Dirac points

$$S = 2$$

Berry curvature of the skyrmion bands

Hamiltonian: $\hat{H} = \int_{\mathrm{BZ}} \frac{d\mathbf{k}}{(2\pi)^2} \Psi_{\mathbf{k}}^{\dagger} \mathcal{H}_{\mathbf{k}} \Psi_{\mathbf{k}}$

$N - dim$ matrix

$\Psi_{\mathbf{k}}^{\dagger} = (|\mathbf{k}, 1\rangle, |\mathbf{k}, 2\rangle \ \ldots \ |\mathbf{k}, N\rangle)$ (Harper equations)

Berry phase: $\gamma_n(\mathcal{C}) = \int_{\mathcal{C}} d\mathbf{k} \cdot \underline{\boldsymbol{\mathcal{A}}_n(\mathbf{k})}$

$\boldsymbol{\mathcal{A}}_n(\mathbf{k}) = i \langle \Psi_{\mathbf{k},n} | \nabla_{\mathbf{k}} | \Psi_{\mathbf{k},n} \rangle$

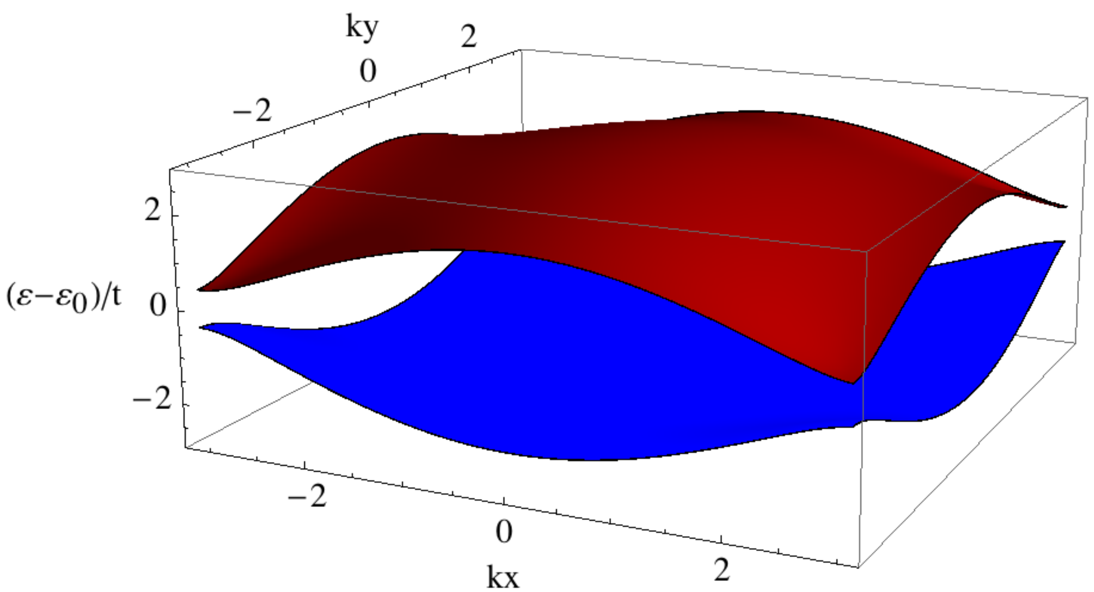
Berry connection

Gauge-invariant property: Berry curvature

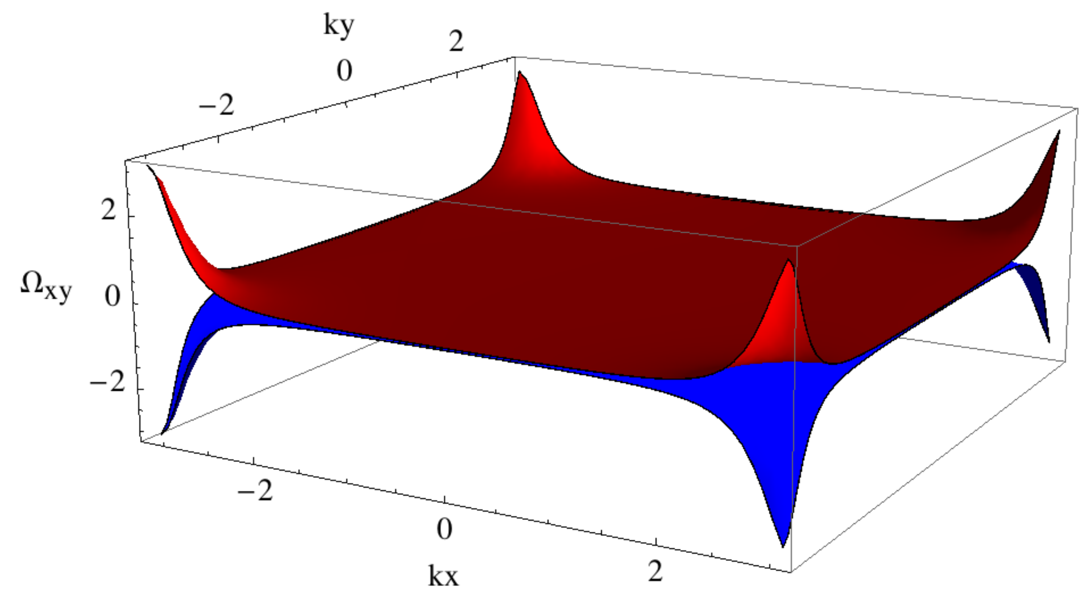
$$\boldsymbol{\Omega}_n(\mathbf{k}) = \nabla_{\mathbf{k}} \times \boldsymbol{\mathcal{A}}_n(\mathbf{k})$$

Example: S = 1

Skyrmion bands



Berry curvature

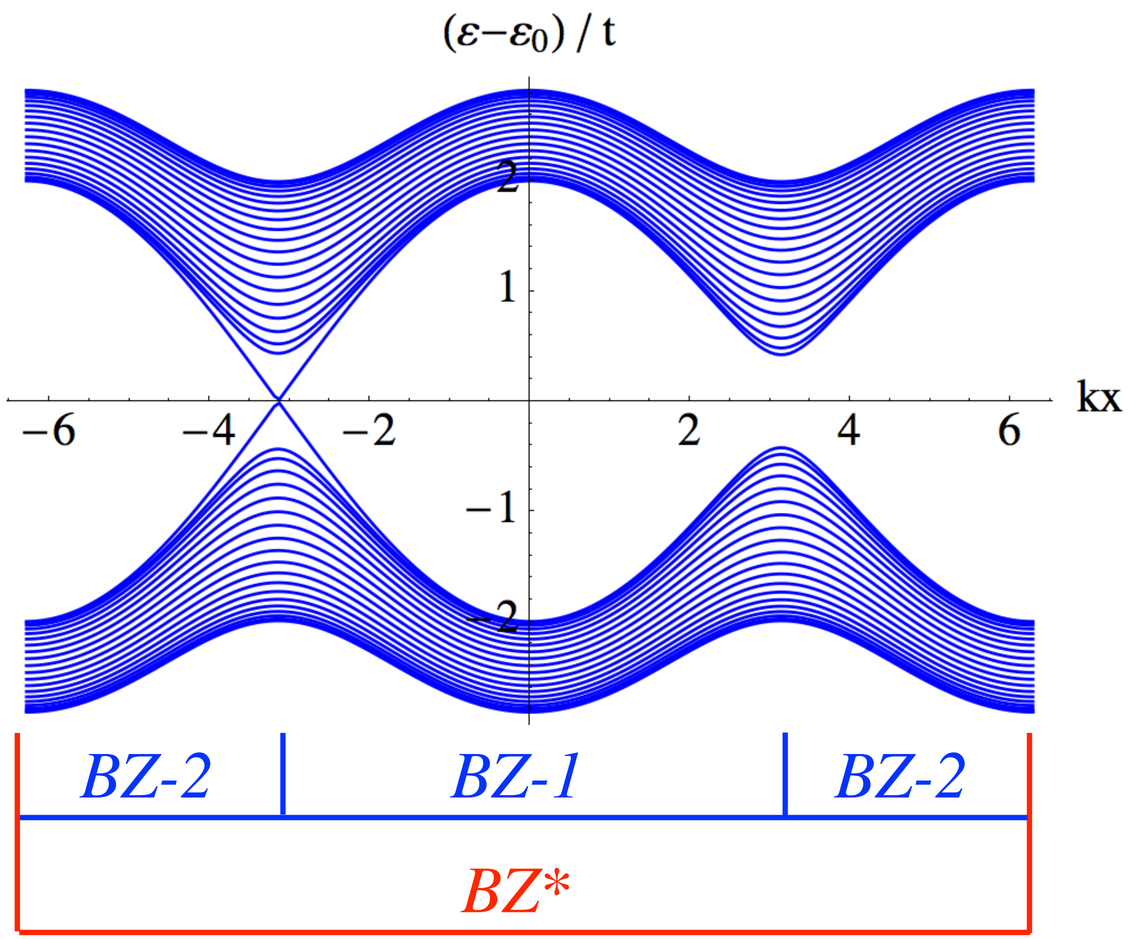


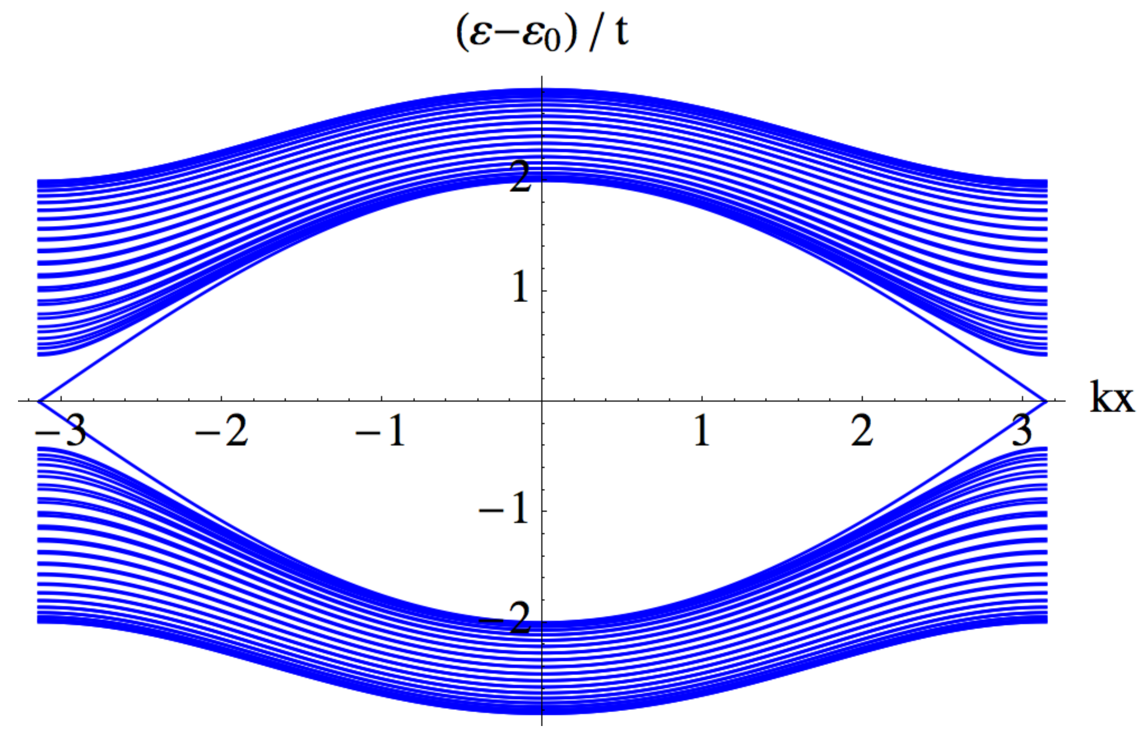
$$\mathcal{Q} = -1,\ t' = -0.1t$$

Chern number: $C_{\pm} = \int_{\underline{\mathrm{BZ}^*}} \frac{d\mathbf{k}}{2\pi}\, \Omega^{\pm}_{xy}(\mathbf{k}) = \pm 1$

periodic conditions!

Edge states $S=1$





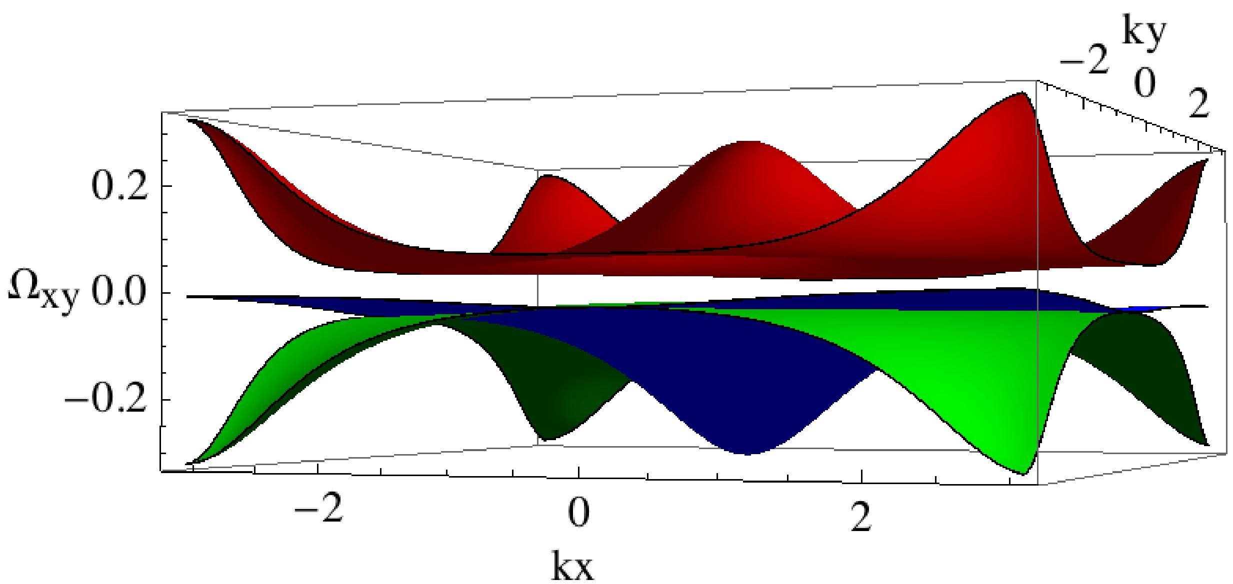
reduced zone (after folding)

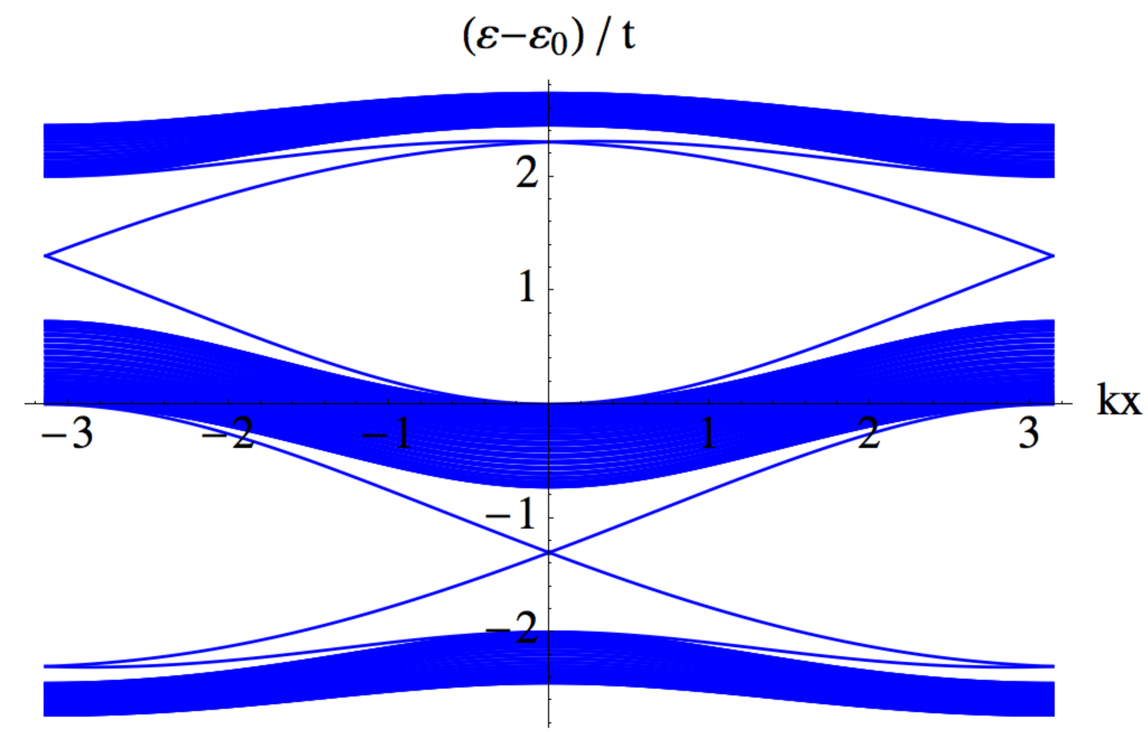
Edge-states wf: over-complete (non-orthogonal) set

Skyrmion edge-states are deformed and naturally hybridized with other modes

Example: $S=3/2$

Berry curvatures





Highest and lowest energy bands: $C = \text{sign}\,(\mathcal{Q})$

Remaining band: $C = -2\,\text{sign}\,(\mathcal{Q})$

spectrum in a strip
(after folding)

Skyrmion bands: general features

- Umklapp processes open gaps at the BZ edges
- The gaps are controlled by the size of the texture compared with the lattice spacing, $V \sim (a/R)^{3/2}$
- The bands become flatter as the spin quantum number increases
- The Chern number of lowest/highest energy bands is $C = \mathbf{sign}\,(\mathcal{Q})$
- The Berry curvature of the lowest energy bands become uniform:

$$\Omega_{xy} \sim \frac{A_c}{4\pi S \mathcal{Q}} \quad \text{for large} \quad S$$

Wave-packet dynamics

Equation of motion of a wp centered at $(\boldsymbol{r}, \boldsymbol{k})$:

$$\hbar \dot{\boldsymbol{k}} = \boldsymbol{F}$$

classical perturbation

$$\dot{\boldsymbol{r}} = \frac{1}{\hbar} \frac{\partial \varepsilon_{\boldsymbol{k}}}{\partial \boldsymbol{k}} + \underline{\Omega_{xy} \left(\boldsymbol{k} \right) \hat{\boldsymbol{z}} \times \dot{\boldsymbol{k}}}$$

anomalous velocity

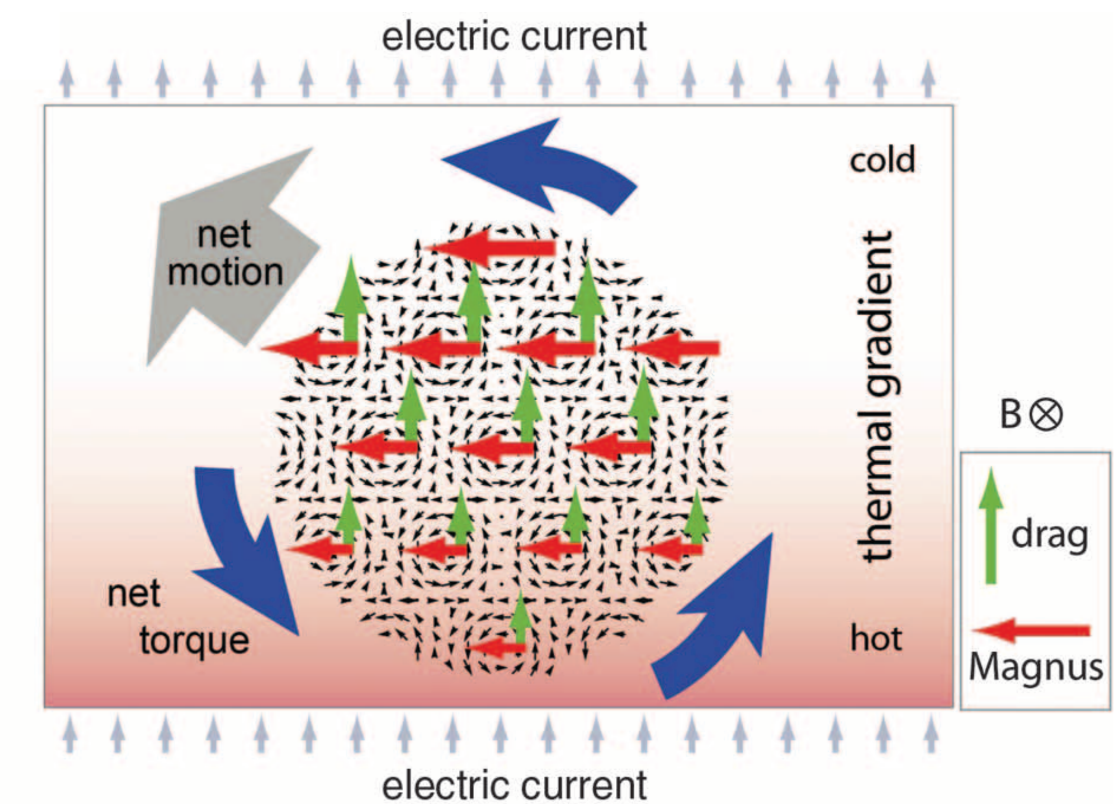
Classical limit: $\lim_{S \to \infty} \frac{1}{\hbar} \frac{\partial \varepsilon_{\boldsymbol{k}}}{\partial \boldsymbol{k}} = 0, \quad \lim_{S \to \infty} \frac{1}{\hbar} \Omega_{xy} \left(\boldsymbol{k} \right) = \frac{1}{4 \pi s Q}$

$$4 \pi s Q \, \dot{\boldsymbol{r}} \times \hat{\boldsymbol{z}} = \boldsymbol{F}$$

classical equation of motion

Local vs transport (divergence free) currents

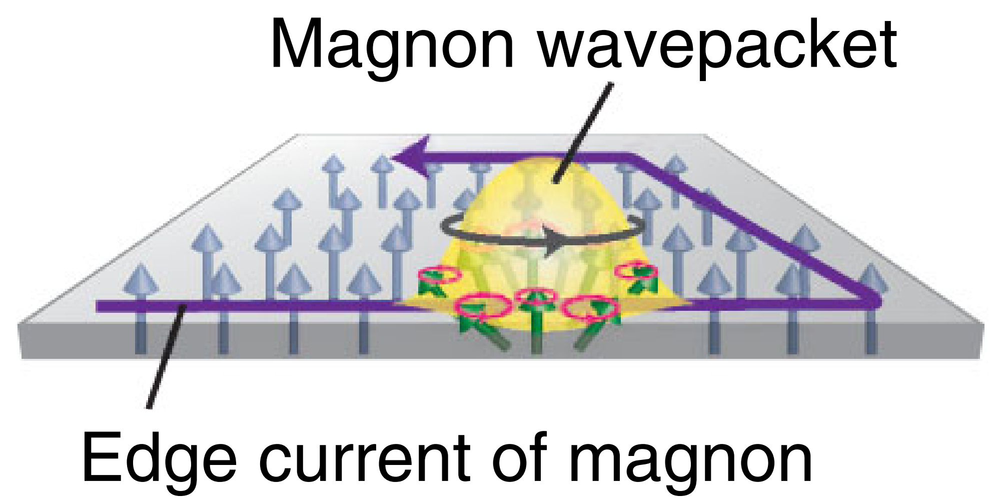
Classically:



Jonietz *et al.*, Science **330**, 1648 (2010)

dissipative + reactive (linear and angular momentum transfer) forces

Semi-classically:



Matsumoto & Murakami, PRL **106**, 197202 (2011)

wp self-rotation provided by the orbital moment of skyrmion bands

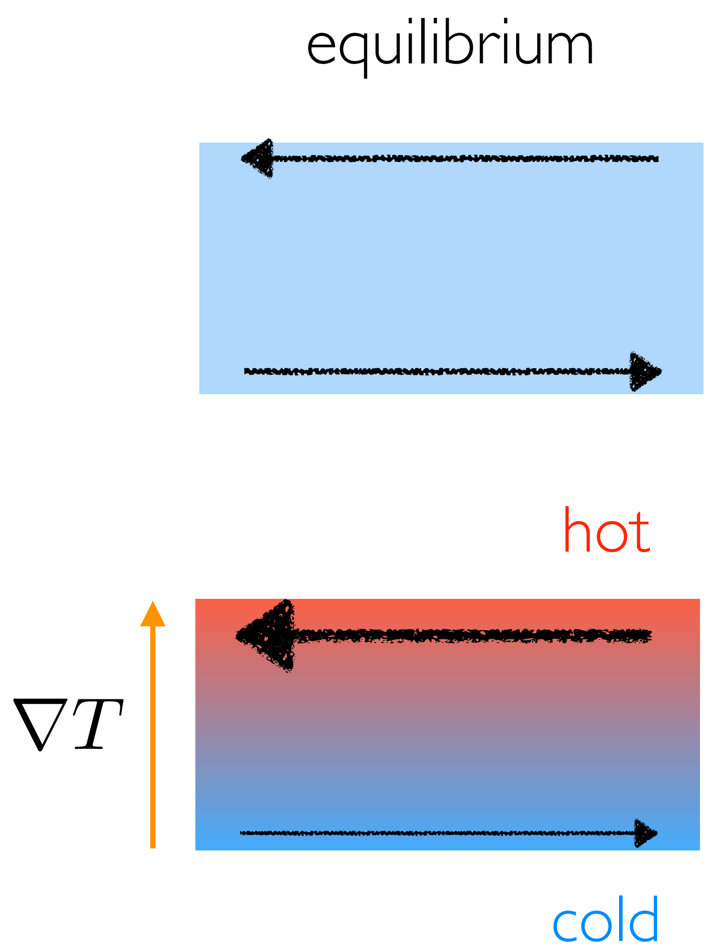
+

persistent edge currents (even at thermal equilibrium)

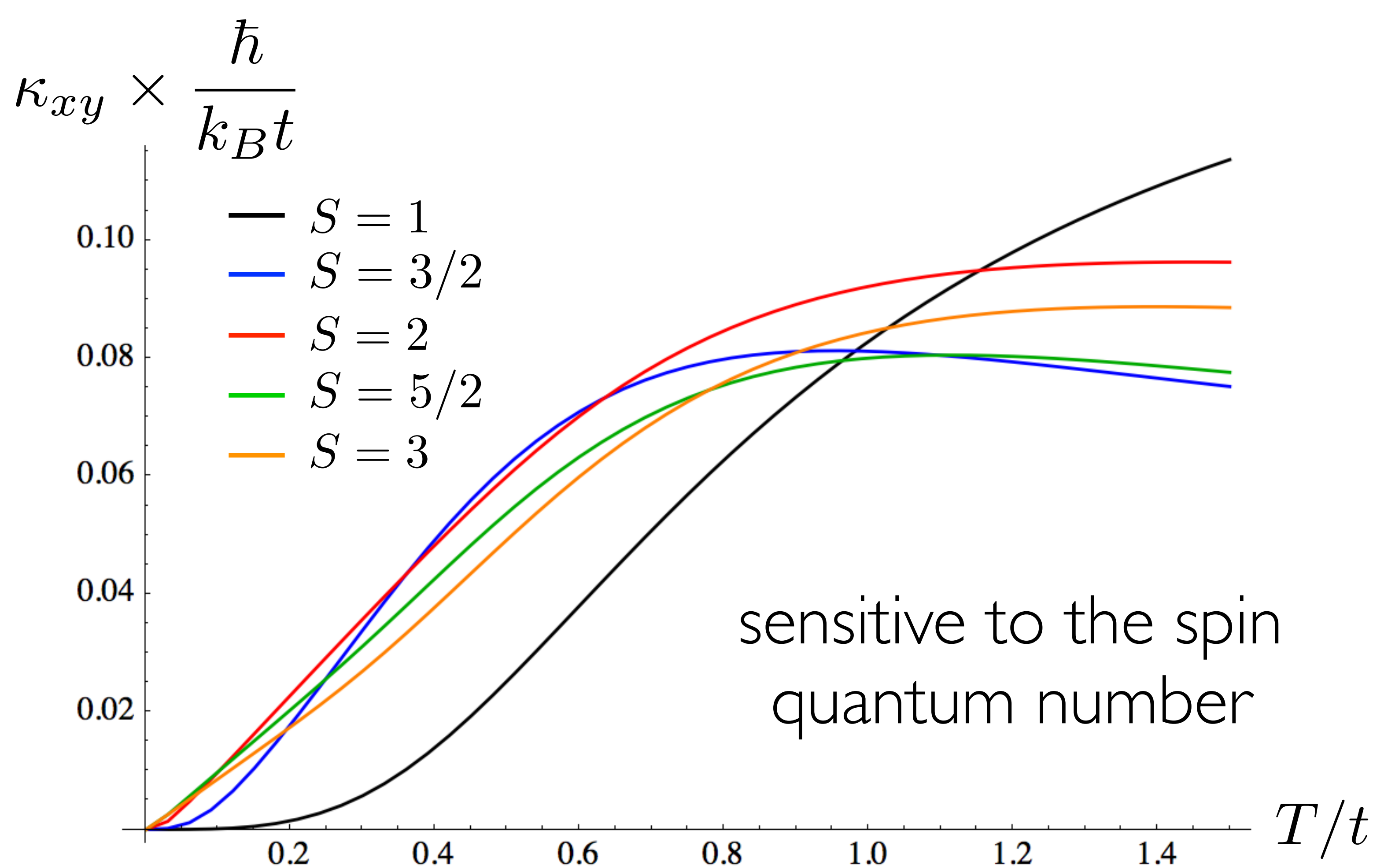
Thermal Hall effect

$$\kappa_{xy} = -\frac{k_B^2 T}{\hbar} \sum_n \int \frac{d\mathbf{k}}{(2\pi)^2} \underline{\eta\left(f^{(0)}_{n,\mathbf{k}}\right)} \Omega^n_{xy}(\mathbf{k})$$

subtracting the divergence-free component



$$J^Q_x = \kappa_{xy}\left(-\partial_y T\right)$$



sensitive to the spin quantum number

Take-home messages

- Quantum skyrmions are local excitations (coherent superposition of magnon-bound states) with boson statistics (in an electrically insulating film, i.e., no coupling with fermions).
- The spectrum is arranged in $N = 2S\,|\mathcal{Q}|$ bands.
- The bands are characterized by non-trivial Berry curvatures, the quantum descendant of the Magnus force
- Skyrmion wave-packets mimic the dynamics of classical textures. When the skyrmion size is comparable to the lattice spacing, the Hall response depends on the spin quantum number.

DE-SC0012190

Wave functions and operator ordering*

- In this limit, the Hilbert space is isomorphic to the space of analytic functions of $z = (x \mp iy)/\ell_N$ (lowest LL of the monopole field)

- The eigenfunctions are in 1-to-1 correspondence, but they are not the same:

$$\Psi_n(\mathbf{r}) \equiv \langle \mathbf{r} | LL = 0;\, n \rangle \longrightarrow \frac{r^n r^{in\theta} e^{-\frac{r^2}{2\ell_N^2}}}{\sqrt{\pi n!}\,\ell_N}$$

- Nevertheless, the normalization densities are properly related:

$$d^2\mathbf{r}\; |\Psi_n(\mathbf{r})|^2 \longrightarrow \frac{dz^* dz}{2\pi i}\, e^{-z^* z}\; |\Psi_n(z)|^2$$

- Anti-normal order: *positions* $(\hat{a}^\dagger)$ to the right

*Like in the QHE: Girvin & Jach, PRB **29**, 5617 (1984)